

A sufficient condition for a polynomial to be stable.

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Abstract

A real polynomial is called Hurwitz (stable) if all its zeros have negative real parts. For a given $n \in \mathbb{N}$ we find the smallest possible constant $d_n > 0$ such that if the coefficients of $F(z) = a_0 + a_1z + \dots + a_nz^n$ are positive and satisfy the inequalities $a_k a_{k+1} > d_n a_{k-1} a_{k+2}$ for $k = 1, 2, \dots, n-2$, then $F(z)$ is Hurwitz.

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1 Introduction and statement of results.

A real polynomial F is called Hurwitz (stable) if all its zeros have negative real parts, i.e. $F(z_0) = 0 \Rightarrow \operatorname{Re} z_0 < 0$. Polynomial stability problems of various types arise in a number of problems in mathematics and engineering. We refer to [5, Chapter 15] or [11, Chapter 9] for deep surveys on the stability theory.

The following statement (usually attributed to A. Stodola, see, for example, [12]) is the well-known necessary condition for a real polynomial to be stable.

Statement A. $F(z) = a_0 + a_1z + \dots + a_nz^n \in \mathbb{R}[z]$, $a_n > 0$, is stable $\Rightarrow a_j > 0$, $0 \leq j \leq n - 1$.

The following famous theorem gives the necessary and sufficient conditions for a polynomial to be stable.

The Routh-Hurwitz Criterion (see, for example, [5, pp. 225-230]).
The polynomial $F(z) = a_0 + a_1z + \dots + a_nz^n$, $a_n > 0$, is stable if and only if the first n principal minors of the corresponding Hurwitz matrix

$$H(F) := \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & \dots & 0 \\ 0 & a_n & a_{n-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

are positive.

Note that the verification of positivity of principal minors is, in general, a very difficult problem. In [3] T. Craven and G. Csordas obtained the following useful and easily verified condition of positivity of all minors of a matrix.

Theorem A. Denote by \tilde{c} the unique real root of $x^3 - 5x^2 + 4x - 1 = 0$ ($\tilde{c} \approx 4.0796$). Let $M = (a_{ij})$ be an $n \times n$ matrix with the properties

- (a) $a_{ij} > 0$ ($1 \leq i, j \leq n$) and
- (b) $a_{ij}a_{i+1,j+1} \geq \tilde{c} a_{i,j+1}a_{i+1,j}$ ($1 \leq i, j \leq n - 1$).

Then all minors of M are positive.

Using Theorem A and continuity reasonings D.K. Dimitrov and J.M. Peña proved the following theorem.

Theorem B [4]. Let \tilde{c} be defined as in Theorem A. If the coefficients of $F(z) = a_0 + a_1z + \dots + a_nz^n$ are positive and satisfy the inequalities

$$a_k a_{k+1} \geq \tilde{c} a_{k-1} a_{k+2} \quad \text{for } k = 1, 2, \dots, n - 2,$$

then $F(z)$ is Hurwitz. In particular, the conclusion is true if

$$a_k^2 \geq \sqrt{\tilde{c}} a_{k-1} a_{k+1} \quad \text{for } k = 1, 2, \dots, n - 1.$$

In [8] the authors of this note have proved that Theorem A remains valid if one replace the constant \tilde{c} by the constant $c_n := 4 \cos^2 \frac{\pi}{n+1}$. In [8] it is also shown that in the statement of Theorem A the constant c_n is the smallest

possible not only in the class of matrices with positive entries but in the classes of Toeplitz matrices and of Hankel matrices. We recall that a matrix M is Toeplitz matrix if it is of the form $M = (a_{j-i})$ and a matrix M is Hankel matrix if it is of the form $M = (a_{j+i})$. In this paper we will show that the constant c_n is not the smallest possible in the class of Hurwitz matrices.

The natural problem is: for a given $n \in \mathbb{N}$ what is the smallest possible constant d_n such that if the coefficients of $F(z) = a_0 + a_1z + \dots + a_nz^n$ are positive and satisfy the inequalities $a_k a_{k+1} > d_n a_{k-1} a_{k+2}$ for $k = 1, 2, \dots, n-2$, then $F(z)$ is Hurwitz? Our main result is the following theorem which solves this problem.

Theorem 1. *Let x_0 be the (unique) positive root of the polynomial $x^3 - x^2 - 2x - 1$ ($x_0 \approx 2.1479$).*

1. *If the coefficients of $F(z) = \sum_{k=0}^4 a_k z^k$ are positive and satisfy the inequalities $a_k a_{k+1} > 2a_{k-1} a_{k+2}$ for $k = 1, 2$, then $F(z)$ is Hurwitz. In particular, the conclusion is true if $a_k^2 > \sqrt{2} a_{k-1} a_{k+1}$ for $k = 1, 2, 3$.*

2. *If the coefficients of $F(z) = \sum_{k=0}^5 a_k z^k$ are positive and satisfy the inequalities $a_k a_{k+1} > x_0 a_{k-1} a_{k+2}$ for $k = 1, 2, 3$, then $F(z)$ is Hurwitz. In particular, the conclusion is true if $a_k^2 > \sqrt{x_0} a_{k-1} a_{k+1}$ for $k = 1, 2, 3, 4$.*

3. *If the coefficients of $F(z) = \sum_{k=0}^n a_k z^k$, $n > 5$, are positive and satisfy the inequalities $a_k a_{k+1} \geq x_0 a_{k-1} a_{k+2}$ for $k = 1, 2, \dots, n-2$, then $F(z)$ is Hurwitz. In particular, the conclusion is true if $a_k^2 \geq \sqrt{x_0} a_{k-1} a_{k+1}$ for $k = 1, 2, \dots, n-1$.*

Note that

$$\frac{a_k a_{k+1}}{a_{k-1} a_{k+2}} = \frac{a_k^2}{a_{k-1} a_{k+1}} \frac{a_{k+1}^2}{a_k a_{k+2}},$$

and thus the following theorem demonstrates that the constants in Theorem 1 are the smallest possible for every n .

Theorem 2.

1. *For every $d \leq \sqrt{2}$ there exists a polynomial $F(z) = \sum_{k=0}^4 a_k z^k$ with positive coefficients under condition $a_k^2 = d a_{k-1} a_{k+1}$ for $k = 1, 2, 3$, such that $F(z)$ is not Hurwitz.*

2. *For every $d \leq \sqrt{x_0}$ there exists a polynomial $F(z) = \sum_{k=0}^5 a_k z^k$ with positive coefficients under condition $a_k^2 = d a_{k-1} a_{k+1}$ for $k = 1, 2, 3, 4$, such that $F(z)$ is not Hurwitz.*

3. *For every $n > 5$ and every $\varepsilon > 0$ there exists a polynomial $F(z) = \sum_{k=0}^n a_k z^k$ with positive coefficients under condition $a_k^2 > (\sqrt{x_0} - \varepsilon) a_{k-1} a_{k+1}$ for $k = 1, 2, \dots, n-1$, such that $F(z)$ is not Hurwitz.*

Theorem 1 may be generalized for entire functions as follows.

Theorem 3. *If the coefficients of $G(z) = \sum_{k=0}^{\infty} a_k z^k$ are positive and satisfy the inequalities $a_k a_{k+1} \geq x_0 a_{k-1} a_{k+2}$ for $k \in \mathbb{N}$, then all zeros of $G(z)$ have negative real parts. In particular, the conclusion is true if $a_k^2 \geq \sqrt{x_0} a_{k-1} a_{k+1}$ for $k \in \mathbb{N}$.*

As we will show in the proof of Theorem 2 the constant in Theorem 3 is the smallest possible.

To prove Theorem 1 we use the famous Hermite-Biehler Criterion. The following statement is a version of the Hermite-Biehler theorem.

The Hermite-Biehler Criterion of stability. (see [2] and [6], or [10, Ch. VII]). *Let $F(z) = \sum_{k=0}^n a_k z^k$ be a polynomial with positive coefficients. The polynomial F is stable if and only if the following two polynomials $f(z) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m a_{2m} z^m$ and $g(z) = z \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} (-1)^m a_{2m+1} z^m$ have simple real interlacing zeros.*

We will use also the following result by Hutchinson [7, p. 327].

Theorem C. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $\forall k$. Inequality $a_n^2 \geq 4a_{n-1}a_{n+1}$, $\forall n \geq 1$, holds if and only if the following two properties hold:*

- (i) *The zeros of $f(z)$ are all real, simple and negative and*
- (ii) *the zeros of any polynomial $\sum_{k=m}^n a_k z^k$, formed by taking any number of consecutive terms of $f(z)$, are all real and non-positive.*

Theorem C is closely connected with the above mention sufficient condition for a matrix to have positive minors. About the connection between the property of a polynomial to have only real non-positive zeros and the positivity of minors of certain matrix see, for example, [1].

2 Proof of Theorems 1 and 3.

It is well-known (and easy verified) that polynomials of degree 1 and 2 with positive leading coefficient are stable if and only if all their coefficients are positive.

Let $F(z) = a_0 + a_1 z + \dots + a_n z^n$ be a polynomial with positive coefficients. Denote by

$$s_j = \frac{a_j a_{j+1}}{a_{j-1} a_{j+2}}, \quad 1 \leq j \leq n-2. \quad (1)$$

Let $F(z) = \sum_{j=0}^3 a_j z^j$ be a polynomial with positive coefficients. Polynomial F is stable if and only if $s_1 > 1$ (see, for example, [12, p. 34]). Really, for $F(z) = \sum_{j=0}^3 a_j z^j$ we have $f(z) = a_0 - a_2 z$ and $g(z) = z(a_1 - a_3 z)$. Both

polynomials have simple real zeros, and these zeros interlace if and only if $0 < \frac{a_0}{a_2} < \frac{a_1}{a_3} \Leftrightarrow s_1 > 1$.

Proof of Theorem 1. To prove Theorem 1 we will use the Hermite-Biehler Criterion of stability. Let us prove the statement 1 of Theorem 1.

For $F(z) = \sum_{j=0}^4 a_j z^j$ we have $f(z) = a_0 - a_2 z + a_4 z^2$ and $g(z) = z(a_1 - a_3 z)$. Using our notations we can express two zeros of the polynomial $f(z)$ in such a way:

$$t_{1,2} = \frac{a_0}{a_2} \frac{s_1 s_2}{2} \left(1 \mp \sqrt{1 - \frac{4}{s_1 s_2}} \right).$$

These zeros are real and distinct since $\min(s_1, s_2) > 2$. Two zeros of the polynomial $g(z)$ are $t_0^* = 0$, $t_1^* = \frac{a_0}{a_2} s_1$, they are real and distinct. The polynomial $F(z)$ is stable if and only if

$$0 < \frac{a_0}{a_2} \frac{s_1 s_2}{2} \left(1 - \sqrt{1 - \frac{4}{s_1 s_2}} \right) < \frac{a_0}{a_2} s_1 < \frac{a_0}{a_2} \frac{s_1 s_2}{2} \left(1 + \sqrt{1 - \frac{4}{s_1 s_2}} \right). \quad (2)$$

The first inequality in (2) obviously holds.

The second inequality in (2) is equivalent to $s_2 \left(1 - \sqrt{1 - \frac{4}{s_1 s_2}} \right) < 2$. The left-hand side of this inequality is strictly decreasing in s_1 , so this inequality follows from $s_2 \left(1 - \sqrt{1 - \frac{2}{s_2}} \right) \leq 2$ (we paste $s_1 = 2$). The left-hand side of the last inequality is strictly decreasing in s_2 , and for $s_2 = 2$ the left-hand side is equal to the right-hand side. By these reasons the second inequality in (2) holds.

The third inequality in (2) is equivalent to $s_2 \left(1 + \sqrt{1 - \frac{4}{s_1 s_2}} \right) > 2$. Since $\min(s_1, s_2) > 2$ the last inequality is true.

Thus, polynomial $F(z) = \sum_{j=0}^4 a_j z^j$ with positive coefficients is stable provided $\min(s_1, s_2) > 2$.

Let us prove the statement 2 of Theorem 1. For $F(z) = \sum_{j=0}^5 a_j z^j$ we have $f(z) = a_0 - a_2 z + a_4 z^2$ and $g(z) = z(a_1 - a_3 z + a_5 z^2)$. Using our notations we can express two zeros of the polynomial $f(z)$ in such a way:

$$t_{1,2} = \frac{a_0}{a_2} \frac{s_1 s_2}{2} \left(1 \mp \sqrt{1 - \frac{4}{s_1 s_2}} \right).$$

These zeros are real and distinct since $\min(s_1, s_2) > x_0 > 2$. The polynomial $g(z)$ has three distinct real zeros which can be written in such a way:

$$t_0^* = 0, \quad t_1^* = \frac{a_0}{a_2} \frac{s_1 s_2 s_3}{2} \left(1 - \sqrt{1 - \frac{4}{s_2 s_3}} \right), \quad t_2^* = \frac{a_0}{a_2} \frac{s_1 s_2 s_3}{2} \left(1 + \sqrt{1 - \frac{4}{s_2 s_3}} \right).$$

The polynomial $F(z)$ is stable if and only if

$$0 < \frac{a_0 s_1 s_2}{a_2} \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{s_1 s_2}}\right) < \frac{a_0 s_1 s_2 s_3}{a_2} \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{s_2 s_3}}\right) < \quad (3)$$

$$\frac{a_0 s_1 s_2}{a_2} \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{s_1 s_2}}\right) < \frac{a_0 s_1 s_2 s_3}{a_2} \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{s_2 s_3}}\right).$$

The first inequality in (3) obviously holds.

The second inequality in (3) is equivalent to $\left(1 - \sqrt{1 - \frac{4}{s_1 s_2}}\right) < s_3 \left(1 - \sqrt{1 - \frac{4}{s_2 s_3}}\right)$. The left-hand side of this inequality is strictly decreasing in s_1 , and $s_1 > x_0 > 2$ so this inequality follows from $\left(1 - \sqrt{1 - \frac{2}{s_2}}\right) \leq s_3 \left(1 - \sqrt{1 - \frac{4}{s_2 s_3}}\right)$ (we paste $s_1 = 2$). The right-hand side of the last inequality is strictly decreasing in s_3 and $\lim_{s_3 \rightarrow \infty} s_3 \left(1 - \sqrt{1 - \frac{4}{s_2 s_3}}\right) = \frac{2}{s_2}$, so the second inequality in (3) follows from the obvious one $\left(1 - \sqrt{1 - \frac{2}{s_2}}\right) \leq \frac{2}{s_2}$. Thus, the second inequality in (3) is true.

Let us check that under our assumptions the third inequality in (3) holds, or, equivalently $s_3 \left(1 - \sqrt{1 - \frac{4}{s_2 s_3}}\right) < \left(1 + \sqrt{1 - \frac{4}{s_1 s_2}}\right)$. The left-hand side of this inequality is strictly decreasing in s_3 , the right-hand side is strictly increasing in s_1 , so this inequality follows from $x_0 \left(1 - \sqrt{1 - \frac{4}{s_2 x_0}}\right) \leq \left(1 + \sqrt{1 - \frac{4}{x_0 s_2}}\right)$, or, equivalently, $(1 + x_0) \sqrt{1 - \frac{4}{s_2 x_0}} \geq x_0 - 1$. The left-hand side of the last inequality is strictly increasing in s_2 , and the last inequality follows from $(1 + x_0) \sqrt{1 - \frac{4}{x_0^2}} \geq x_0 - 1$ or, equivalently, $x_0^3 - x_0^2 - 2x_0 - 1 \geq 0$. By the definition of x_0 this is true. Thus, the third inequality in (3) holds.

Let us check that under our assumptions the fourth inequality in (3) holds, or, equivalently $1 + \sqrt{1 - \frac{4}{s_1 s_2}} < s_3 \left(1 + \sqrt{1 - \frac{4}{s_2 s_3}}\right)$. The left-hand side is strictly increasing in s_1 and $\lim_{s_1 \rightarrow \infty} 1 + \sqrt{1 - \frac{4}{s_1 s_2}} = 2$, so the last inequality follows from $2 \leq s_3 \left(1 + \sqrt{1 - \frac{4}{s_2 s_3}}\right)$. The right-hand side is strictly increasing in $s_2 > x_0 > 2$, so the last inequality follows from the obvious inequality $2 \leq s_3 \left(1 + \sqrt{1 - \frac{2}{s_3}}\right)$. So, the fourth inequality in (3) holds.

Thus, polynomial $F(z) = \sum_{j=0}^5 a_j z^j$ with positive coefficients is stable provided $\min(s_1, s_2, s_3) > x_0$.

Remark. It follows from the proof of the statement 2 of Theorem 1 that if $\min(s_1, s_2, s_3) \geq x_0$ then $t_0^* < t_1 < t_1^* \leq t_2 < t_2^*$ (the notation is the same as in the proof of statement 2 of Theorem 1).

Let $F(z) = \sum_{k=0}^n a_k z^k$, $a_k > 0$, be a polynomial satisfying the condition $s_j \geq x_0$ for $j = 1, 2, \dots, n-2$. Without loss of generality we can assume that $a_0 = 1$. Denote by

$$p_j = \frac{a_{j-1}}{a_j}, \quad j = 1, 2, \dots, n. \quad (4)$$

Using this notation we can write

$$F(z) = 1 + a_1 z + a_2 z^2 + \dots + a_n z^n = 1 + \frac{z}{p_1} + \frac{z^2}{p_1 p_2} + \dots + \frac{z^n}{p_1 p_2 \dots p_n}.$$

To prove the statement 3 of Theorem 1 we need some lemmas. The statement below is the direct consequence of the statements 1 and 2 of Theorem 1 and Remark.

Lemma 1. Let $\tilde{f}(z) = 1 - \frac{z}{p_1 p_2} + \frac{z^2}{p_1 p_2 p_3 p_4}$, $\tilde{g}(z) = 1 - \frac{z}{p_2 p_3} + \frac{z^2}{p_2 p_3 p_4 p_5}$, where $p_j > 0, 1 \leq j \leq 5$. Suppose that $p_{j+2}/p_j \geq x_0$, where x_0 is equal to the unique positive root of the polynomial $x^3 - x^2 - 2x - 1$. Denote by $0 < x_1 < x_2$ the zeros of \tilde{f} , $0 < x_1^* < x_2^*$ the zeros of \tilde{g} . Then

$$\tilde{g}(x_2) \leq 0, \quad \tilde{f}(x_1^*) \leq 0. \quad (5)$$

If $\hat{f} = -1 + \frac{z}{p_3 p_4}$, $\hat{g} = 1 - \frac{z}{p_2 p_3}$, then

$$\hat{g}(x_2) < 0, \quad \hat{f}(x_1^*) < 0. \quad (6)$$

We need the following Lemma.

Lemma 2. Let

$$F(z) = 1 - \frac{z}{\rho_1} + \frac{z^2}{\rho_1 \rho_2} - \dots + (-1)^n \frac{z^n}{\rho_1 \rho_2 \dots \rho_n}, \quad n \geq 3, \quad (7)$$

be a polynomial with $\rho_j > 0$ and $\min\{\frac{\rho_{j+1}}{\rho_j}, 1 \leq j \leq n-1\} > 1$. Denote by

$$R_j(z, F) = \begin{cases} 1 - \frac{z}{\rho_j} + \frac{z^2}{\rho_j \rho_{j+1}}, & \text{if } j = 1, 2, \dots, n-1; \\ -1 + \frac{z}{\rho_1}, & \text{if } j = 0; \\ 1 - \frac{z}{\rho_n}, & \text{if } j = n. \end{cases} \quad (8)$$

Then

$$(-1)^j F(x) > -K_j(x)R_j(x, F), \quad \rho_{j-2} < x < \rho_{j+3}, \quad j = 0, 1, \dots, n, \quad (9)$$

where $K_j(x) > 0$, $\rho_j = 0$ for $j < 1$, and $\rho_j = \infty$ for $j > n$.

Proof of Lemma 2. Let us fix $j \in \{1, 2, \dots, n-1\}$. We have

$$\begin{aligned} (-1)^j F(x) &= \left((-1)^j + \sum_{k=1}^{j-2} (-1)^{k+j} \frac{x^k}{\rho_1 \rho_2 \cdots \rho_k} \right) + \\ & \left[-\frac{x^{j-1}}{\rho_1 \rho_2 \cdots \rho_{j-1}} + \frac{x^j}{\rho_1 \rho_2 \cdots \rho_j} - \frac{x^{j+1}}{\rho_1 \rho_2 \cdots \rho_{j+1}} \right] + \sum_{k=j+2}^n (-1)^{j+k} \frac{x^k}{\rho_1 \rho_2 \cdots \rho_k} =: \\ & \Sigma_1(x) + \left[-\frac{x^{j-1}}{\rho_1 \rho_2 \cdots \rho_{j-1}} + \frac{x^j}{\rho_1 \rho_2 \cdots \rho_j} - \frac{x^{j+1}}{\rho_1 \rho_2 \cdots \rho_{j+1}} \right] + \Sigma_2(x). \end{aligned} \quad (10)$$

For every $x \in (\rho_{j-2}, \rho_{j+3})$ $\frac{x^k}{\rho_1 \rho_2 \cdots \rho_k} < \frac{x^{k+1}}{\rho_1 \rho_2 \cdots \rho_{k+1}}$ holds for $0 \leq k \leq j-3$. So for all $x \in (\rho_{j-2}, \rho_{j+3})$ summands in $\Sigma_1(x)$ are alternating in sign and their moduli are increasing. Analogously for all $x \in (\rho_{j-2}, \rho_{j+3})$ summands in $\Sigma_2(x)$ are alternating in sign and their moduli are decreasing. So $\Sigma_1(x) \geq 0$, $\Sigma_2(x) \geq 0$ for all $x \in (\rho_{j-2}, \rho_{j+3})$, so for $j = 1, 2, \dots, n-1$

$$(-1)^j F(x) \geq -\frac{x^{j-1}}{\rho_1 \rho_2 \cdots \rho_{j-1}} + \frac{x^j}{\rho_1 \rho_2 \cdots \rho_j} - \frac{x^{j+1}}{\rho_1 \rho_2 \cdots \rho_{j+1}}, \quad x \in (\rho_{j-2}, \rho_{j+3}). \quad (11)$$

Thus (9) is proved for $j = 1, 2, \dots, n-1$.

For $j = 0$ (9) follows from the inequality

$$F(x) = -\left(-1 + \frac{x}{\rho_1}\right) + \sum_{k=2}^n (-1)^k \frac{x^k}{\rho_1 \rho_2 \cdots \rho_k} > -R_0(x, F), \quad 0 < x < \rho_3.$$

We use the fact that for $0 < x < \rho_3$ the summands in $\sum_{k=2}^n (-1)^k \frac{x^k}{\rho_1 \rho_2 \cdots \rho_k}$ are alternating in sign and their moduli are decreasing. So the sign of this sum coincides with the sign of the first summand (for $k = 2$) and this sign is positive.

For $j = n$ (9) follows from the inequality

$$\begin{aligned} (-1)^n F(x) &= -\frac{x^{n-1}}{\rho_1 \rho_2 \cdots \rho_{n-1}} \left(1 - \frac{x}{\rho_n}\right) + (-1)^n + \\ & \sum_{k=1}^{n-2} (-1)^{k+n} \frac{x^k}{\rho_1 \rho_2 \cdots \rho_k} > -\frac{x^{n-1}}{\rho_1 \rho_2 \cdots \rho_{n-1}} R_n(x, F), \quad \rho_{n-2} < x < \infty. \end{aligned}$$

We use the fact that for $\rho_{n-2} < x < \infty$ the summands in the expression $\left((-1)^n + \sum_{k=1}^{n-2} (-1)^{k+n} \frac{x^k}{\rho_1 \rho_2 \dots \rho_k}\right)$ are alternating in sign and their moduli are increasing. So the sign of this expression coincides with the sign of the last summand (for $k = n - 2$) and this sign is positive. \square

Lemma 3. *Let $F(z) = 1 - \frac{z}{\rho_1} + \frac{z^2}{\rho_1 \rho_2} - \dots + (-1)^n \frac{z^n}{\rho_1 \rho_2 \dots \rho_n}$, $n \geq 3$, be a polynomial with $\rho_j > 0$ and $\min\{\frac{\rho_{j+1}}{\rho_j}, 1 \leq j \leq n - 1\} > 4$. Let polynomials $R_j(z, F)$, $j = 0, 1, \dots, n$, be defined by (8).*

1. *Polynomial $R_j(z)$, $j = 1, 2, \dots, n - 1$, has simple real zeros, which we denote by $0 < \omega_1(j) < \omega_2(j)$.*
2. *Polynomial $F(z)$ has simple real zeros $0 < x_1 < x_2 < \dots < x_n$.*
3. *$\omega_1(j) > \rho_j$, $\omega_2(j) < \rho_{j+1}$.*
4. *$\sqrt{\rho_{j-1} \rho_j} < x_j < \sqrt{\rho_j \rho_{j+1}}$, $j = 1, 2, \dots, n$, (where we put $\rho_0 = 1$, $\rho_{n+1} = +\infty$).*
5. *$x_j < \omega_1(j)$, $x_{j+1} > \omega_2(j)$, $j = 1, 2, \dots, n - 1$.*

Proof of Lemma 3. Since $\min\{\frac{\rho_{j+1}}{\rho_j}, 1 \leq j \leq n - 1\} > 4$ then

$$R_j(\sqrt{\rho_j \rho_{j+1}}, F) = 2 - \sqrt{\frac{\rho_{j+1}}{\rho_j}} < 0, \quad 1 \leq j \leq n - 1.$$

The statements 1 and 3 of Lemma immediately follow from this. The fact that $\min\{\frac{\rho_{j+1}}{\rho_j}, 1 \leq j \leq n - 1\} > 4$ implies that polynomial $F(z)$ has simple real zeros is the well known (see, for example, [9]). This fact and statement 4 of Lemma follow from the statements:

$$\begin{aligned} F(\sqrt{\rho_0 \rho_1}) > 0, \quad -F(\sqrt{\rho_1 \rho_2}) > 0, \quad F(\sqrt{\rho_2 \rho_3}) > 0, \quad \dots, \\ (-1)^{n-1} F(\sqrt{\rho_{n-1} \rho_n}) > 0, \quad \lim_{x \rightarrow \infty} [(-1)^n F(x)] = +\infty. \end{aligned} \quad (12)$$

The last statement is obvious, the rest of inequalities is the direct consequence of (9). Since for all $z \in (\omega_1(j), \omega_2(j))$ we have $R_j(z, F) < 0$, $j = 1, 2, \dots, n - 1$, than the statement 5 is the direct consequence of (9). \square

Let us prove now the statement 3 of Theorem 1.

By the Hermite-Biehler criterion the polynomial F is stable if and only if the following two polynomials

$$f(z) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m a_{2m} z^m$$

and

$$g(z) = z \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} (-1)^m a_{2m+1} z^m$$

have simple real interlacing zeros.

Since

$$\frac{a_{2m}^2}{a_{2m-2}a_{2m+2}} = \frac{a_{2m-1}a_{2m}}{a_{2m-2}a_{2m+1}} \frac{a_{2m}a_{2m+1}}{a_{2m-1}a_{2m+2}} \geq x_0^2 > 4, \quad m = 1, 2, \dots, \lfloor \frac{n-2}{2} \rfloor,$$

and

$$\frac{a_{2m+1}^2}{a_{2m-1}a_{2m+3}} = \frac{a_{2m}a_{2m+1}}{a_{2m-1}a_{2m+2}} \frac{a_{2m+1}a_{2m+2}}{a_{2m}a_{2m+3}} \geq x_0^2 > 4, \quad m = 1, 2, \dots, \lfloor \frac{n-3}{2} \rfloor,$$

polynomials $f(z)$ and $g(z)$ have simple real zeros. It remains to prove that under our assumptions zeros of polynomials $f(z)$ and $g(z)$ are interlacing.

We need the following notations. Let P be a real polynomial. Denote by $N_{(a,b)}(P)$ the number of zeros of P in the interval (a,b) . Denote by $0 < t_1 < t_2 < \dots < t_{\lfloor \frac{n}{2} \rfloor}$ zeros of $f(z)$ and by $0 = t_0^* < t_1^* < t_2^* < \dots < t_{\lfloor \frac{n-1}{2} \rfloor}^*$ zeros of $g(z)$. We obtain the fact that zeros of polynomials $f(z)$ and $g(z)$ are interlacing as a consequence of the following lemma.

Lemma 4.

$$N_{(t_j, t_{j+1})}(g) \geq 1, \quad j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1. \quad (13)$$

$$N_{(t_j^*, t_{j+1}^*)}(f) \geq 1, \quad j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor - 1. \quad (14)$$

Proof of Lemma 4. For $F(z) = 1 + \frac{z}{p_1} + \frac{z^2}{p_1 p_2} + \dots + \frac{z^n}{p_1 p_2 \dots p_n}$ we have $f(z) = 1 - \frac{z}{p_1 p_2} + \frac{z^2}{p_1 p_2 p_3 p_4} + \dots$ and $g(z) = \frac{z}{p_1} (1 - \frac{z}{p_2 p_3} + \frac{z^2}{p_2 p_3 p_4 p_5} + \dots)$. Put $g_1(z) = g(z) p_1 / z$. Note that the polynomial f has the form (7) with $\rho_j = p_{2j-1} p_{2j}$ and the polynomial g_1 has the form (7) with $\rho_j = p_{2j} p_{2j+1}$. We will consider the polynomials $R_j(z, f)$ for $j = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ and $R_j(z, g_1)$ for $j = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$, these polynomials are defined by (8).

At first we will prove (13) with $n \geq 7$. By Lemma 3 we have $t_j < \omega_1(j)$, $t_{j+1} > \omega_2(j)$, where $\omega_1(j), \omega_2(j)$ are zeros of $R_j(z, f)$. To prove (13) it is sufficient to prove that

$$(-1)^{j-1} g_1(\omega_1(j)) > 0, \quad j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1, \quad (15)$$

$$(-1)^{j-1}g_1(\omega_2(j)) < 0, \quad j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1. \quad (16)$$

By (9) we have

$$(-1)^{j-1}g_1(t) > -K_{j-1}(t)R_{j-1}(t, g_1), \quad (17)$$

where $p_{2j-6}p_{2j-5} < t < p_{2j+4}p_{2j+5}$, $j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$;

$$(-1)^j g_1(t) > -K_j(t)R_j(t, g_1), \quad (18)$$

where $p_{2j-4}p_{2j-3} < t < p_{2j+6}p_{2j+7}$, $j = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1$. In (17) and (18) the functions $K_j(t)$ are positive. By the statement 3 of Lemma 3 we have $p_{2j-1}p_{2j} < \omega_1(j) < \omega_2(j) < p_{2j+1}p_{2j+2}$, so inequalities (17) and (18) are valid for $t \in (\omega_1(j), \omega_2(j))$, $j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1$. In (17) we apply Lemma 1 with $\tilde{f}(t) = R_{j-1}(t, g_1)$, $j = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor - 1$, $\hat{f}(t) = R_0(t, g_1)$ and $\tilde{g}(t) = R_j(t, f)$, $j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1$. We have

$$R_j(\omega_1(j), g_1) < 0, \quad j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1. \quad (19)$$

From (17), (19) we obtain (15).

In (18) we apply Lemma 1 with $\tilde{f}(t) = R_j(t, f)$ and $\tilde{g}(t) = R_j(t, g_1)$, $j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$. If n is an odd number, then $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. If n is an even number we also apply Lemma 1 with $\hat{g}(t) = R_{\lfloor \frac{n}{2} \rfloor - 1}(t, g_1)$ and $\tilde{f}(t) = R_{\lfloor \frac{n}{2} \rfloor - 1}(t, f)$. We have

$$R_j(\omega_2(j), g_1) < 0, \quad j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1. \quad (20)$$

From (18), (20) we obtain (16).

The statement (14) with $n \geq 6$ can be proved analogously. By Lemma 3 we have $t_j^* < \omega_1^*(j)$, $t_{j+1}^* > \omega_2^*(j)$, where $\omega_1^*(j), \omega_2^*(j)$ are zeros of $R_j(z, g_1)$. To prove (14) it is sufficient to prove that

$$(-1)^{j-1}f(\omega_1^*(j)) < 0, \quad j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor - 1, \quad (21)$$

$$(-1)^{j-1}f(\omega_2^*(j)) > 0, \quad j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor - 1. \quad (22)$$

By (9) we have

$$(-1)^j f(t) > -K_j(t)R_j(t, f), \quad (23)$$

where $p_{2j-5}p_{2j-4} < t < p_{2j+5}p_{2j+6}$, $j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1$ and

$$(-1)^{j+1}f(t) > -K_{j+1}(t)R_{j+1}(t, f), \quad (24)$$

where $p_{2j-3}p_{2j-2} < t < p_{2j+7}p_{2j+8}$, $j = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor - 1$. In (23) and (24) the functions $K_j(t)$ are positive. By statement 3 of Lemma 3 we have $p_{2j}p_{2j+1} < \omega_1^*(j) < \omega_2^*(j) < p_{2j+2}p_{2j+3}$, so inequalities (23) and (24) are valid for $t \in (\omega_1^*(j), \omega_2^*(j))$, $j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor - 1$. Using (23), (24) and Lemma 1 (analogously to the proof of (15) and (16)) we can obtain (21) and (22). Thus, (14) is proved.

Applying (21) and (22) for $n = 6$ we have

$$f(\omega_1^*) < 0, \quad f(\omega_2^*) > 0,$$

where ω_1^*, ω_2^* are zeros of the polynomial $R_1(z, g_1) = g_1(z)$. Besides this

$$f(0) = 1 > 0, \quad \lim_{t \rightarrow \infty} f(t) = -\infty.$$

Thus, (13) is proved for $n = 6$. \square

So, zeros of polynomials $f(z)$ and $g(z)$ are interlacing. Applying (13) for $j = 1$ we have

$$g_1(\omega_1(1)) > 0,$$

where $\omega_1(1)$ is the smallest zero of the polynomial $R_1(z, f)$. Besides this $g_1(0) = 1 > 0$. Thus, $t_1 < \omega_1(1) < t_1^*$.

Theorem 1 is proved. \square

Proof of Theorem 3. To prove Theorem 3 we use just the same reasonings as in the proof of Theorem 1. Instead of the Hermite-Biehler Criterion of stability for polynomials we will apply the following generalization of the Hermite-Biehler theorem on entire functions of order not greater than 1 and minimal type of growth.

Theorem HB [10, Chapter 7]. *Let $G(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, be an entire function of order not greater than 1 and minimal type of growth. All zeros of G have negative real parts if and only if the following two entire functions $f(z) = \sum_0^{\infty} (-1)^m a_{2m} z^m$ and $g(z) = z \sum_0^{\infty} (-1)^m a_{2m+1} z^m$ have simple real interlacing zeros.*

Let us consider a function $G(z) = \sum_{k=0}^{\infty} a_k z^k$ with positive coefficients under condition $a_k a_{k+1} \geq x_0 a_{k-1} a_{k+2}$ for $k \in \mathbb{N}$. Note that

$$\frac{a_{2k+1}}{a_{2k+2}} \cdot \frac{a_2}{a_1} = \frac{a_{2k+1} a_{2k}}{a_{2k+2} a_{2k-1}} \cdot \frac{a_{2k-1} a_{2k-2}}{a_{2k} a_{2k-3}} \dots \frac{a_3 a_2}{a_4 a_1} \geq x_0^k, \quad k \geq 1,$$

and

$$\frac{a_{2k}}{a_{2k+1}} \cdot \frac{a_1}{a_0} = \frac{a_{2k} a_{2k-1}}{a_{2k+1} a_{2k-2}} \cdot \frac{a_{2k-2} a_{2k-3}}{a_{2k-1} a_{2k-4}} \dots \frac{a_2 a_1}{a_3 a_0} \geq x_0^k, \quad k \geq 1.$$

So we have

$$a_{2k+2} \leq \frac{1}{x_0^k} \cdot \frac{a_2}{a_1} \cdot a_{2k+1}, \quad a_{2k+1} \leq \frac{1}{x_0^k} \cdot \frac{a_1}{a_0} \cdot a_{2k}, \quad k \geq 1.$$

Whence

$$\begin{aligned} a_{2k+2} &\leq \frac{1}{x_0^k} \cdot \frac{a_2}{a_1} \cdot a_{2k+1} \leq \frac{1}{x_0^{2k}} \cdot \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} \cdot a_{2k} \leq \\ &\frac{1}{x_0^{2k+2(k-1)}} \cdot \left(\frac{a_2}{a_0}\right)^2 a_{2k-2} \leq \dots \leq \frac{1}{x_0^{k(k+1)}} \cdot \left(\frac{a_2}{a_0}\right)^k a_2, \quad k \geq 1. \end{aligned}$$

Analogously

$$\begin{aligned} a_{2k+1} &\leq \frac{1}{x_0^k} \cdot \frac{a_1}{a_0} \cdot a_{2k} \leq \frac{1}{x_0^{2k-1}} \cdot \frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot a_{2k-1} \leq \\ &\frac{1}{x_0^{k-1+2k-3}} \cdot \left(\frac{a_2}{a_0}\right)^2 a_{2k-3} \leq \dots \leq \frac{1}{x_0^{k^2}} \cdot \left(\frac{a_2}{a_0}\right)^k a_1, \quad k \geq 1. \end{aligned}$$

It is well-known that the order of an entire function with coefficients a_k is given by the expression $\limsup_{n \rightarrow \infty} \frac{n \log n}{\log |a_n|^{-1}}$ (see, for example [10, Chapter 1]). So, G is an entire function of order 0.

It is easy to prove that statements of Lemmas 2, 3, 4 remain true for the entire function G . It follows from this that Theorem 3 is true.

Theorem 3 is proved. \square

3 Proof of Theorem 2.

Let us prove the statement 1 of Theorem 2. We consider a polynomial $T_\beta(z) = \sum_{j=1}^4 a_j(\beta)z^j = 1 + \beta^3 z + \beta^4 z^2 + \beta^3 z^3 + z^4, \beta > 0$, and note that for this polynomial $s_j(\beta) = \frac{a_j(\beta)a_{j+1}(\beta)}{a_{j-1}(\beta)a_{j+2}(\beta)} = \beta^4, j = 1, 2$. We have $T_\beta(z) = z^2((z^2 + z^{-2}) + \beta^3(z + z^{-1}) + \beta^4)$. Denote by $w = z + z^{-1}$ and note that $Re z < 0 \Leftrightarrow Re w < 0$. We have $T_\beta(z) = z^2(w^2 + \beta^3 w + \beta^4 - 2)$. It is obvious that quadratic real polynomial is stable if and only if all its coefficients have the same sign. So $T_\beta(z)$ is stable if and only if $\beta^4 - 2 > 0$. In other words, if $s_1(\beta) = s_2(\beta) = \beta^4 \leq 2$ then polynomial $T_\beta(z)$ is not stable.

To prove the statement 2 of Theorem 2 we consider a polynomial $M_\beta(z) = \sum_{j=1}^5 b_j(\beta)z^j = 1 + \beta^4 z + \beta^6 z^2 + \beta^6 z^3 + \beta^4 z^4 + z^5, \beta > 0$, and note that for

this polynomial $s_j(\beta) = \frac{b_j(\beta)b_{j+1}(\beta)}{b_{j-1}(\beta)b_{j+2}(\beta)} = \beta^4$, $j = 1, 2, 3$. We have

$$\begin{aligned} M_\beta(z) &= (z+1)(z^4 + (\beta^4 - 1)z^3 + (\beta^6 - \beta^4 + 1)z^2 + (\beta^4 - 1)z + 1) = \\ &= (z+1)z^2((z^2 + z^{-2}) + (\beta^4 - 1)(z + z^{-1}) + (\beta^6 - \beta^4 + 1)) \end{aligned}$$

Using the notation $w = z + z^{-1}$ we can write

$$M_\beta(z) = (z+1)z^2(w^2 + (\beta^4 - 1)w + (\beta^6 - \beta^4 - 1)).$$

Since $\text{sign}(\text{Re } z) = \text{sign}(\text{Re } w)$, $M_\beta(z)$ is stable if and only if $\beta^6 - \beta^4 - 1 > 0$. Note that $(\beta^6 - \beta^4 - 1)(\beta^6 + \beta^4 + 1) = \beta^{12} - \beta^8 - 2\beta^4 - 1$ and polynomial $\beta^6 + \beta^4 + 1$ has no positive zeros. Thus $M_\beta(z)$ is stable if and only if $\beta^4 > x_0$, where x_0 is the unique positive root of the polynomial $x^3 - x^2 - 2x - 1$, and for $\beta^4 < x_0$ the polynomial $M_\beta(z)$ has zeros with positive real parts. So if $s_1(\beta) = s_2(\beta) = s_3(\beta) = \beta^4 < x_0$, then polynomial $M_\beta(z)$ has zeros with positive real parts thus it is not stable.

Let us prove the statement 3 of Theorem 2. Obviously, for every $\varepsilon > 0$ we can choose β in such a way that $x_0 - \varepsilon < \beta^4 < x_0$. So, the polynomial $M_\beta(z)$ has zeros with positive real parts. For every $\varepsilon \in (0, x_0)$ we denote by $\delta = (x_0 - \varepsilon/2)^{\frac{1}{4}}$, so $\delta > 0$, $\delta^4 > x_0 - \varepsilon$. For $n = 6$ we put $Q_{\gamma_1,6}(z) = M_\delta(z) + \gamma_1 z^6$, $\gamma_1 > 0$. Since $M_\delta(z)$ has zeros with positive real parts, $Q_{\gamma_1,6}(z)$ has zeros with positive real parts for γ_1 being small enough. For the polynomial $Q_{\gamma_1,6}(z)$ we have $s_1 = s_2 = s_3 = \delta^4 > x_0 - \varepsilon$, and $s_4 = \frac{1}{\delta^2 \gamma_1} > x_0 - \varepsilon$ for γ_1 being small enough. For γ_1 chosen below and $n = 7$ we put $Q_{\gamma_2,7}(z) = Q_{\gamma_1,6}(z) + \gamma_2 z^7$, $\gamma_2 > 0$. Since $Q_{\gamma_1,6}(z)$ has zeros with positive real parts, $Q_{\gamma_2,7}(z)$ has zeros with positive real parts for γ_2 being small enough. For the polynomial $Q_{\gamma_2,7}(z)$ we have $s_1 = s_2 = s_3 = \delta^4 > x_0 - \varepsilon$, $s_4 = \frac{1}{\delta^2 \gamma_1} > x_0 - \varepsilon$ and $s_5 = \frac{\gamma_1}{\delta^4 \gamma_2} > x_0 - \varepsilon$ for γ_2 being small enough. Reasoning analogously we can construct the example needed for every $n \geq 5$ and example of an entire function.

Theorem 2 is proved. \square

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