On the stability of Taylor sections of a function $\sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}, \ a > 1$.

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July 9, 2008

Abstract

We investigate the following problem: given a positive integer n, which are the smallest values of the constants s_n, such that the zeros of $f_{a,n}(z) := \sum_{k=0}^{n} \frac{z^k}{a^{k^2}}$ are with negative real parts when $a > s_n$?

2000 Mathematics Subject Classification 30C15; 30D15; 26C10; 34D99.

1 Introduction and statement of results.

A real polynomial F is called Hurwitz (stable) polynomial if all its zeros have negative real parts: $F(z_0) = 0 \Rightarrow \text{Re } z_0 < 0$. Polynomial stability problems of various types arise in a number of problems in mathematics and engineering. We refer to [5, Chapter 15] or [13, Chapter 9] for deep surveys on the stability theory.

The following statement (usually attributed to A. Stodola, see, for example, [17]) is the well-known necessary condition for a real polynomial to be stable.

Statement A. $F(z) = a_0 + a_1 z + \ldots + a_n z^n \in \mathbf{R}[z], \ a_n > 0,$ *is stable* $\Rightarrow a_j > 0, \ 0 \leq j \leq n-1.$

The following famous theorem gives the necessary and sufficient conditions for a polynomial to be stable.

The Routh-Hurwitz Criterion (see, for example, [5, pp. 225-230]). *The polynomial* $F(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad a_n > 0$, *is stable if and only if the first* n *principal minors of the corresponding Hurwitz matrix*

$$H(F) := \left\| \begin{matrix} a_{n-1} & a_{n-3} & a_{n-5} & \ldots & 0 \\ a_n & a_{n-2} & a_{n-4} & \ldots & 0 \\ 0 & a_{n-1} & a_{n-3} & \ldots & 0 \\ 0 & a_n & a_{n-2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \right\|$$

are positive.

Note that the verification of positivity of principal minors is, in general, a very difficult problem. Surely, it is not difficult to calculate the determinant of a given matrix with numerical entries. But if the order of a matrix or the entries of a matrix depend on some parameters then the testing of positivity of principal minors is complicated. In [3] T. Craven and G. Csordas obtained the useful and easily verified sufficient condition of positivity of all minors of a matrix. To formulate this condition we need the following definition.

A matrix M is said to be totally positive, if all minors of M are non-negative. About this notion and its applications see [1, 8] and the references therein. A matrix M is said to be strictly totally positive, if all minors of M are strictly positive.

In [3] the following theorem was proved

Theorem A. *Denote by* $\tilde{c}$ *the unique real root of* $x^3 - 5x^2 + 4x - 1 = 0$ *(*$\tilde{c} \approx 4.0796$*). Let* $M = (a_{ij})$ *be an* $n \times n$ *matrix with the properties*
(a) $a_{ij} > 0$ $(1 \le i, j \le n)$ *and*
(b) $a_{ij} a_{i+1,j+1} \ge \tilde{c}\ a_{i,j+1} a_{i+1,j}$ $(1 \le i, j \le n-1)$.
Then M *is strictly totally positive.*

Theorem A provides a convenient sufficient condition for strict total positivity of a matrix with positive entries. Using Theorem A and continuity reasonings D.K. Dimitrov and J.M. Peña proved the following theorem.

Theorem B [4]. *Let* $\tilde{c}$ *be defined as in Theorem A. If the coefficients of* $F(z) = a_0 + a_1 z + \ldots + a_n z^n$ *are positive and satisfy the inequalities*

$$a_k a_{k+1} \ge \tilde{c} a_{k-1} a_{k+2} \quad \textit{for} \quad k = 1, 2, \ldots, n-2, \tag{1}$$

then $F(z)$ *is a Hurwitz polynomial. In particular, the conclusion is true if*

$$a_k^2 \ge \sqrt{\tilde{c}} a_{k-1} a_{k+1} \quad \textit{for} \quad k = 1, 2, \ldots, n-1. \tag{2}$$

In [10] the authors of this note have proved that Theorem A remains valid if one replace the constant $\tilde{c}$ by the constant $c_n := 4\cos^2 \frac{\pi}{n+1}$. In [10] it is also shown that in the statement of Theorem A the constant c_n is the smallest possible not only in the class of matrices with positive entries but in the classes of Toeplitz matrices and of Hankel matrices. We recall that a matrix M is Toeplitz matrix if it is of the form $M = (a_{j-i})$ and a matrix M is Hankel matrix if it is of the form $M = (a_{j+i})$.

In [11] the authors of this note have found the smallest possible constants in the inequalities of the type (1) and (2) such that the statement of Theorem B remains valid.

Theorem C. *Let x_0 be the (unique) positive root of the polynomial x^3-x^2-2x-1 ($x_0 \approx 2.1479$).*
1. If the coefficients of $F(z) = \sum_{k=0}^4 a_k z^k$ are positive and satisfy the inequalities $a_k a_{k+1} > 2a_{k-1}a_{k+2}$ for $k = 1, 2$, then $F(z)$ is a Hurwitz polynomial. In particular, the conclusion is true if $a_k^2 > \sqrt{2}a_{k-1}a_{k+1}$ for $k = 1, 2, 3$.
2. If the coefficients of $F(z) = \sum_{k=0}^5 a_k z^k$ are positive and satisfy the inequalities $a_k a_{k+1} > x_0 a_{k-1}a_{k+2}$ for $k = 1, 2, 3$, then $F(z)$ is a Hurwitz polynomial. In particular, the conclusion is true if $a_k^2 > \sqrt{x_0}a_{k-1}a_{k+1}$ for $k = 1, 2, 3, 4$.
3. If the coefficients of $F(z) = \sum_{k=0}^n a_k z^k$, $n > 5$, are positive and satisfy the inequalities $a_k a_{k+1} \geq x_0 a_{k-1}a_{k+2}$ for $k = 1, 2, \dots, n-2$, then $F(z)$ is a Hurwitz polynomial. In particular, the conclusion is true if $a_k^2 \geq \sqrt{x_0}a_{k-1}a_{k+1}$ for $k = 1, 2, \dots, n-1$.

Note that $\frac{a_k a_{k+1}}{a_{k-1}a_{k+2}} = \frac{a_k^2}{a_{k-1}a_{k+1}} \frac{a_{k+1}^2}{a_k a_{k+2}}$ and thus the following theorem demonstrates that the constants in Theorem C are the smallest possible for every n.

Theorem D.
1. For every $d \leq \sqrt{2}$ there exists a polynomial $F(z) = \sum_{k=0}^4 a_k z^k$ with positive coefficients under condition $a_k^2 = d a_{k-1}a_{k+1}$ for $k = 1, 2, 3$, such that $F(z)$ is not a Hurwitz polynomial.
2. For every $d \leq \sqrt{x_0}$ there exists a polynomial $F(z) = \sum_{k=0}^5 a_k z^k$ with positive coefficients under condition $a_k^2 = d a_{k-1}a_{k+1}$ for $k = 1, 2, 3, 4$, such that $F(z)$ is not a Hurwitz polynomial.
3. For every $n > 5$ and every $\varepsilon > 0$ there exists a polynomial $F(z) = \sum_{k=0}^n a_k z^k$ with positive coefficients under condition $a_k^2 > (\sqrt{x_0} - \varepsilon)a_{k-1}a_{k+1}$ for $k = 1, 2, \dots, n-1$, such that $F(z)$ is not a Hurwitz polynomial.

The extremal entire function for which the inequalities (1) and (2) reduce to equalities is

$$f_a(z) := \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}, \ a > 1. \tag{3}$$

In [4] the following problem was posed: given a positive integer n, which are the smallest values of the constants s_n, such that the zeros of $f_{a,n}(z) := \sum_{k=0}^n \frac{z^k}{a^{k^2}}$ are with negative real parts when $a > s_n$? Further we will explain that for every positive integer n there exists the constant s_n such that all zeros of $f_{a,n}(z)$ have negative real parts if and only if $a > s_n$. We will explain also that there exists the constant s_∞ such that all zeros of $f_a(z)$ have negative real parts if and only if $a > s_\infty$. The following theorem will answer the question formulated in [4].

Theorem 1.

1. $s_3 \leq s_7 \leq s_{11} \leq \ldots$;

2. $s_5 \geq s_9 \geq s_{13} \geq \ldots$;

3. $\lim_{n\to\infty} s_{4n+3} = \lim_{n\to\infty} s_{4n+1} = s_\infty$

Remark. Using the reasonings close to those in the proof of theorem 1 we can prove that the sequence s_{2n} is converging and $\lim_{n\to\infty} s_{2n} = s_\infty$. We do not include the proof of this fact because it contains rather cumbersome calculation.

To prove Theorem 1 we use the famous Hermite-Biehler Criterion.

The Hermite-Biehler Criterion (see [2] and [6], or [12, Ch. VII]). *Let* $M(z) = N(z) + iR(z)$, *where* N *and* R *are real polynomials. The following are equivalent:*
A. All zeros of $M(z)$ *have positive imaginary parts.*
B. The polynomials N *and* R *have simple real interlacing zeros and* $R^{'}(x_0)N(x_0) - R(x_0)N^{'}(x_0) > 0$ *for some* $x_0 \in \mathbf{R}$.

The following statement is a version of the Hermite-Biehler theorem.

The Hermite-Biehler Criterion of stability. *Let* $F(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial with positive coefficients. The polynomial* F *is stable if and only if the following two polynomials* $P(z) := \sum_{0\leq m\leq \frac{n}{2}} (-1)^m a_{2m} z^m$ *and* $zQ(z) := z\sum_{0\leq m\leq \frac{n-1}{2}} (-1)^m a_{2m+1} z^m$ *have simple real interlacing zeros.*

2 Proof of Theorem 1.

We will consider the entire function

$$g_a(z) = f_a(-z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{a^{k^2}}, \ a > 1, \tag{4}$$

and will study the zero distribution of its Taylor sections. Obviously all zeros of $f_a(z)$ (or its n-th Taylor section) lie in the left half plane if and only if all zeros of $g_a(z)$ (or its n-th Taylor section) lie in the right half plane $\{z : \text{ Re } z > 0\}$. Denote by

$$S_n(x, a) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{a^{k^2}}.$$

Let us consider an odd section

$$S_{2n+1}(x, a) = 1 - \frac{x}{a} + \frac{x^2}{a^4} - \frac{x^3}{a^9} + \ldots - \frac{x^{2n+1}}{a^{4n^2+4n+1}}.$$

For this polynomial two polynomials $P_n(x,a)$ and $Q_n(x,a)$ mentioned in the Hermite-Biehler criterion of stability are

$$P_n(x,a) = 1 - \frac{x^2}{a^4} + \frac{x^4}{a^{16}} - \ldots + (-1)^n \frac{x^{2n}}{a^{4n^2}}$$

and

$$Q_n(x,a) = (-\frac{1}{a}) \left(1 - \frac{x^2}{a^8} + \frac{x^4}{a^{24}} - \ldots + (-1)^n \frac{x^{2n}}{a^{4n^2+4n}}\right).$$

The question is: for which a the polynomials $P_n(x,a)$ and $xQ_n(x,a)$ have simple real interlacing zeros?

Denote by $q = a^4$ and $t = x^2$. Polynomials $P_n(x,a)$ and $Q_n(x,a)$ will pass into

$$\begin{aligned} \tilde{P}_n(t,q) &= 1 - \tfrac{t}{q} + \tfrac{t^2}{q^4} - \ldots + (-1)^n \tfrac{t^n}{q^{n^2}}; \\ \tilde{Q}_n(t,q) &= 1 - \tfrac{t}{q^2} + \tfrac{t^2}{q^6} - \ldots + (-1)^n \tfrac{t^n}{q^{n(n+1)}}. \end{aligned} \tag{5}$$

All zeros of polynomials $P_n(x,a)$ and $Q_n(x,a)$ are real and simple if and only if all zeros of polynomials $\tilde{P}_n(t,q)$ and $\tilde{Q}_n(t,q)$ are real and simple. Suppose all zeros of polynomials $\tilde{P}_n(t,q)$ and $\tilde{Q}_n(t,q)$ are real and simple. Denote by $t_1 < t_2 < \ldots < t_n$ the zeros of $\tilde{P}_n(t,q)$ and by $t_1^* < t_2^* < \ldots < t_n^*$ the zeros of $\tilde{Q}_n(t,q)$. Obviously the zeros of $P_n(x,a)$ and $xQ_n(x,a)$ interlace if and only if $t_1 < t_1^* < t_2 < t_2^* < \ldots < t_n < t_n^*$. We have $\tilde{Q}_n(t,q) = \tilde{P}_n(\frac{t}{q}, q)$, so the polynomial $\tilde{P}_n(t,q)$ has only real simple zeros if and only if the polynomial $\tilde{Q}_n(t,q)$ has. Denote by

$$R_n(y,q) := \tilde{P}_n(q^n y, q) = \tilde{Q}_n(q^{n+1} y, q) = \sum_{k=0}^{n} (-1)^k q^{k(n-k)} y^k. \tag{6}$$

Let $y_1 < y_2 < \ldots < y_n$ be the zeros of $R_n(y,q)$. Then the zeros of $\tilde{P}_n(t,q)$ are $t_k = y_k q^n,\ k = 1,2,\ldots,n$; the zeros of $\tilde{Q}_n(t,q)$ are $t_k^* = y_k q^{n+1},\ k = 1,2,\ldots,n$. So the condition $t_k < t_k^*$ obviously holds for all $k = 1,2,\ldots,n$. The condition $t_k^* < t_k,\ k = 1,2,\ldots,n-1$, could be rewrote in the form

$$\frac{y_{k+1}}{y_k} > q,\ k = 1,2,\ldots,n-1. \tag{7}$$

Thus, all zeros of $S_{2n+1}(x,a)$ lie in the right half plane if and only if all zeros of $R_n(y,q)$ are real and these zeros satisfy (7) (we recall that $q = a^4$).

Now we need some notions and facts concerning linear operators on the space of real polynomials which do not decrease the number of real zeros of polynomial.

Definition 1. *A sequence $\{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a multiplier sequence if, whenever the real polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ has only real zeros the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeros. The class of multiplier sequences we will denote by MS.*

The following famous theorem by G. Pólya and J.Schur gives the complete characterization of multiplier sequences:

Theorem E. (see ([16]), ([15]) or ([12], chapter VIII, sec. 3)) *A sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence if and only if the power series* $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ *converges absolutely in the whole complex plane and the entire function* $\Phi(z)$ *or the entire function* $\Phi(-z)$ *admits the representation*

$$Ce^{\sigma z} z^m \prod_{k=1}^{\infty} (1 + \frac{z}{x_k}), \tag{8}$$

where $C \in \mathbf{R}, \sigma \geq 0, m \in \mathbf{N} \cup \{0\}, 0 < x_k \leq \infty,\ \sum_{k=1}^{\infty} \frac{1}{x_k} < \infty.$

The simple consequence of Theorem E is that the sequence $\{\gamma_0, \gamma_1, \ldots, \gamma_l, 0, 0, \ldots\}$ is a multiplier sequence if and only if the polynomial $P(z) = \sum_{k=0}^{l} \frac{\gamma_k}{k!} z^k$ has only real zeros of the same sign.

For a real polynomial P we will denote by $Z_c(P)$ the number of nonreal zeros of P, counting multiplicities.

Definition 2. *A sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *of real numbers is said to be a complex zero decreasing sequence if*

$$Z_c(\sum_{k=0}^{n} \gamma_k a_k z^k) \leq Z_c(\sum_{k=0}^{n} a_k z^k), \tag{9}$$

for any real polynomial $\sum_{k=0}^{n} a_k z^k$*. The class of complex zero decreasing sequences we will denote by CZDS.*

Obviously, $CZDS \subset MS$. The existence of nontrivial CZDS sequences is a consequence of the following remarkable theorem proved by Laguerre and extended by Pólya (see [14] or [15], pp. 314-321).

Theorem F. *Suppose an entire function* $f(z)$ *can be expressed in the form*

$$f(z) = Cz^m e^{-\alpha z^2 + \beta z} \prod_{k=1}^{\infty} (1 + \frac{z}{x_k}) e^{-\frac{z}{x_k}}, \tag{10}$$

where $m \in \mathbf{N} \cup \{0\}, \quad C, \beta \in \mathbf{R}, \quad \alpha \geq 0$ *and* $0 < x_k \leq \infty,\ \sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$*. Then the sequence* $\{f(k)\}_{k=0}^{\infty}$ *is a complex zero decreasing sequence.*

As it follows from the above theorem,

$$\{a^{-k^2}\}_{k=0}^{\infty} \in CZDS \subset MS,\ a \geq 1. \tag{11}$$

Using (11) we conclude that if the polynomial $S_n(x, a_0) := \sum_{k=0}^{n} \frac{(-1)^k x^k}{a_0^{k^2}}$ has only real zeros then for all $a \geq a_0$ polynomials $S_n(x, a) := \sum_{k=0}^{n} \frac{(-1)^k x^k}{a^{k^2}}$ have only real zeros. Thus,

$$\forall n = 2, 3, 4, \ldots\ \exists r_n > 1 : (S_n(x, a) \text{ has only real zeros } \Leftrightarrow a \geq r_n.) \tag{12}$$

In [9] the following problem was solved: for which $a > 1$ the entire function $g_a(z)$ (and its Taylor sections $S_n(x, a)$, $n = 2, 3, \ldots$) has only real (real and simple) zeros. The following statements were proved which we have united in one theorem now for the convenience of references.

Theorem G. ([9]) *There exists a constant r_∞ ($r_\infty \approx 1.79$) such that:*

1. $\sqrt{3} = r_3 \leq r_5 \leq r_7 \leq \cdots$; $\lim_{k\to\infty} r_{2k+1} = r_\infty$;
2. $2 = r_2 \geq r_4 \geq r_6 \geq \cdots$; $\lim_{k\to\infty} r_{2k} = r_\infty$;
3. *for every $n \geq 2$ the polynomial $S_n(x, r_n)$ has multiple real zeros;*
4. *for every $a > r_n$ the polynomial $S_n(x, a)$ has only simple real zeros;*
5. *for every $n \geq 2$ and $a \geq r_n$ all zeros of $S_n(x, a)$ lie in the interval (a, a^{2n-1});*
6. *for every $n \geq 2$ and $a \geq r_n$ the polynomial $S_n(x, a)$ has exactly two zeros in the interval (a^{2n-3}, a^{2n-1});*
7. $g_a(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{a^{k^2}}$ *has only real zeros* $\Leftrightarrow$ $a \geq r_\infty$;
8. $g_a(z)$ *has only simple real zeros* $\Leftrightarrow$ $a > r_\infty$;
9. *for $a \geq r_\infty$ the function $g_a(z)$ has exactly two zeros in the interval (a, a^3);*
10. *for $a > r_\infty$ the function $g_a(z)$ has exactly n intervals of positivity and n intervals of negativity in the interval (a, a^{4n-1}).*
11. *for $a > r_\infty$ and $j \geq 2$ the following inequality holds: $(-1)^j g_a(a^{2j}) > 0$.*

We will also use the known fact that if $\{\gamma_k\}_{k=0}^{\infty} \in MS$ then whenever the real polynomial $\sum_{k=0}^{n} a_k z^k$ is stable the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ is stable (see, for example, [12, chapter VIII, sec. 3]). In particular, if the polynomial $S_n(x, a_0) := \sum_{k=0}^{n} \frac{(-1)^k x^k}{a_0^{k^2}}$ has all zeros in the right half plane then for all $a \geq a_0$ polynomials $S_n(x, a) := \sum_{k=0}^{n} \frac{(-1)^k x^k}{a^{k^2}}$ have all zeros in the right half plane. Thus,

$$\begin{gathered} \forall n = 1, 2, 3, \ldots \ \exists s_n \geq 0 : \\ (S_n(x, a) \text{ has all zeros in the right half plane } \Leftrightarrow a > s_n.) \end{gathered} \tag{13}$$

By the direct calculation one can obtain that

$$\begin{gathered} (S_1(x, a) \text{ has all zeros in the right half plane } \Leftrightarrow a > 0) \\ \Rightarrow s_1 = 0; \\ (S_2(x, a) \text{ has all zeros in the right half plane } \Leftrightarrow a > 0) \\ \Rightarrow s_2 = 0; \end{gathered} \tag{14}$$

$$(S_3(x,a) \text{ has all zeros in the right half plane } \Leftrightarrow a>1)$$
$$\Rightarrow s_3 = 1;$$
$$(S_4(x,a) \text{ has all zeros in the right half plane } \Leftrightarrow a^4>2)$$
$$\Rightarrow s_4 \approx 1.1892;$$
$$(S_5(x,a) \text{ has all zeros in the right half plane } \Leftrightarrow a^6-a^4-1>0)$$
$$\Rightarrow s_5 \approx 1.2106;$$
$$(S_6(x,a) \text{ has all zeros in the right half plane } \Leftrightarrow a^4>2)$$
$$\Rightarrow s_6 \approx 1.1892;$$
$$(S_7(x,a) \text{ has all zeros in the right half plane } \Leftrightarrow$$
$$a^{16}-2a^{12}+1>0 \wedge a>1) \Rightarrow s_7 \approx 1.1646.$$

We need the following lemma.

Lemma 1. *Suppose for some $q>1$ all zeros of $R_{2n}(y,q)$ are real and these zeros satisfy (7). Then*

1. All zeros of $R_{2n+1}(y,q)$ are real and these zeros satisfy (7).

2. All zeros of $R_{2n+2}(y,q)$ are real and these zeros satisfy (7).

Proof of Lemma 1.

1. The reality of zeros is a consequence of Theorem G (1 and 2). Note that

$$R_{2n+1}(y,q) = \sum_{k=0}^{2n}(-1)^k q^{k(2n+1-k)} y^k - y^{2n+1} = \\ \sum_{k=0}^{2n}(-1)^k q^{k(2n-k)} (qy)^k - y^{2n+1} = R_{2n}(qy,q) - y^{2n+1}, \tag{15}$$

and

$$R_{2n+1}(y,q) = 1 - \sum_{k=1}^{2n+1}(-1)^{k-1} q^{k(2n+1-k)} y^k = \\ 1 - \sum_{k=0}^{2n}(-1)^k q^{(k+1)(2n-k)} y^{k+1} = 1 - q^{2n} y \sum_{k=0}^{2n}(-1)^k q^{k(2n-k)} (\tfrac{y}{q})^k = \\ 1 - q^{2n} y R_{2n}(\tfrac{y}{q},q). \tag{16}$$

Denote by $y_1 < y_2 < \ldots < y_{2n-1} < y_{2n}$ the zeros of $R_{2n}(y,q)$. Note that $R_{2n}(t,q)>0$ for $t \in (0,y_1)\cup(y_2,y_3)\cup\ldots\cup(y_{2n-2},y_{2n-1})\cup(y_{2n},\infty)$ and $R_{2n}(t,q)<0$ for $t \in (y_1,y_2)\cup(y_3,y_4)\cup\ldots\cup(y_{2n-1},y_{2n})$. By (15) we have

$$R_{2n+1}(y,q) < 0, \ \textit{for}\ y_{2k-1} < qy < y_{2k}, \ k=1,2,\ldots,n. \tag{17}$$

And by (16) we have

$$R_{2n+1}(y,q) > 0, \ \textit{for}\ y_{2k-1} < \frac{y}{q} < y_{2k}, \ k=1,2,\ldots,n. \tag{18}$$

Thus, we have

$$R_{2n+1}(0,q) = 1 > 0; \\ R_{2n+1}(y,q) < 0, \ \textit{for}\ \tfrac{y_{2k-1}}{q} < y < \tfrac{y_{2k}}{q}, \ k=1,2,\ldots,n; \\ R_{2n+1}(y,q) > 0, \ \textit{for}\ qy_{2k-1} < y < qy_{2k}, \ k=1,2,\ldots,n; \\ R_{2n+1}(\infty,q) = -\infty. \tag{19}$$

Since by our assumption zeros of $R_{2n}(y,q)$ satisfy (7) we have

$$\frac{y_{2k+1}}{y_{2k-1}} = \frac{y_{2k+1}}{y_{2k}} \cdot \frac{y_{2k}}{y_{2k-1}} > q^2,\ k = 1, 2, \ldots, n-1,$$

whence

$$\frac{y_{2k-1}}{q} < qy_{2k-1} < \frac{y_{2k+1}}{q},\ k = 1, 2, \ldots, n-1,$$

or,

$$\frac{y_1}{q} < \frac{y_2}{q} < qy_1 < qy_2 < \frac{y_3}{q} < \frac{y_4}{q} < qy_3 < qy_4 < \ldots$$
$$< \frac{y_{2n-1}}{q} < \frac{y_{2n}}{q} < qy_{2n-1} < qy_{2n}.$$

Denote by $u_1 < u_2 < \ldots < u_{2n} < u_{2n+1}$ the zeros of $R_{2n+1}(y,q)$. Then

$$u_{2k-1} < \frac{y_{2k-1}}{q} < \frac{y_{2k}}{q} < u_{2k} < qy_{2k-1} < qy_{2k} < u_{2k+1},\ k = 1, 2, \ldots, n. \qquad (20)$$

Thus, $\frac{u_{2k}}{u_{2k-1}} > \frac{y_{2k}}{y_{2k-1}} > q$ and $\frac{u_{2k+1}}{u_{2k}} > \frac{y_{2k}}{y_{2k-1}} > q$.

The statement 1 from Lemma 1 is proved.

Let us prove the statement 2. We recall that $S_n(x,q) = R_n(\frac{x}{q^n}, q)$. So, by Theorem G (5) all zeros of $R_{2n}(y,q)$ lie in the interval (q^{-2n+1}, q^{2n-1}). We have

$$\begin{gathered} R_{2n+2}(y,q) = \textstyle\sum_{k=0}^{2n} (-1)^k q^{k(2n+2-k)} y^k - q^{2n+1} y^{2n+1} + y^{2n+2} = \\ \textstyle\sum_{k=0}^{2n} (-1)^k q^{k(2n-k)} (q^2 y)^k - y^{2n+1}(q^{2n+1} - y) = \\ R_{2n}(q^2 y, q) - y^{2n+1}(q^{2n+1} - y); \end{gathered} \qquad (21)$$

$$\begin{gathered} R_{2n+2}(y,q) = 1 - \textstyle\sum_{k=1}^{2n+1} (-1)^{k-1} q^{k(2n+2-k)} y^k + y^{2n+2} = \\ 1 - \textstyle\sum_{k=0}^{2n} (-1)^k q^{(k+1)(2n+1-k)} y^{k+1} + y^{2n+2} = \\ (1 + y^{2n+2}) - q^{2n+1} y \textstyle\sum_{k=0}^{2n} (-1)^k q^{k(2n-k)} y^k = \\ (1 + y^{2n+2}) - q^{2n+1} y R_{2n}(y,q). \end{gathered} \qquad (22)$$

By (21) we have $R_{2n+2}(y,q) < 0$ for $\frac{y_{2k-1}}{q^2} < y < \frac{y_{2k}}{q^2}$, $k = 1, 2, \ldots, n$ (since $y_{2n} < q^{2n-1} \Leftrightarrow \frac{y_{2n}}{q^2} < q^{2n-3}$, the last summand in (21) is negative).

By (22) we have $R_{2n+2}(y,q) > 0$ for $y_{2k-1} < y < y_{2k}$, $k = 1, 2, \ldots, n$, and $R_{2n+2}(0,q) = 1 > 0$.

We recall that all zeros of $R_{2n}(y,q)$ lie in the interval (q^{-2n+1}, q^{2n-1}).

By (7) we have $\frac{y_{2k+1}}{y_{2k-1}} > q^2$ and thus

$$0 < \frac{y_1}{q^2} < \frac{y_2}{q^2} < y_1 < y_2 < \frac{y_3}{q^2} < \frac{y_4}{q^2} < y_3 < y_4 < \ldots <$$

$$\frac{y_{2n-1}}{q^2} < \frac{y_{2n}}{q^2} < y_{2n-1} < y_{2n} < q^{2n-1}.$$

Denote by $v_1 < v_2 < \ldots < v_{2n+2}$ the zeros of $R_{2n+2}(y,q)$. By Theorem G (6) there are two zeros of $R_{2n+2}(y,q)$ in the interval (q^{2n-1}, q^{2n+1}). Thus we obtain

$$v_{2k-1} < \frac{y_{2k-1}}{q^2} < \frac{y_{2k}}{q^2} < v_{2k} < y_{2k-1} < y_{2k} < v_{2k+1},\ k = 1, 2, \ldots, n,$$

whence

$$\frac{v_{2k}}{v_{2k-1}} > \frac{y_{2k}}{y_{2k-1}} > q, \quad \frac{v_{2k+1}}{v_{2k}} > \frac{y_{2k}}{y_{2k-1}} > q,\ k = 1, 2, \ldots, n.$$

It remains to prove that $\frac{v_{2n+2}}{v_{2n+1}} > q$. Note that $R_{2n+2}(y,q) = y^{2n+2}R_{2n+2}(\frac{1}{y}, q)$, so $v_{2n+2} = \frac{1}{v_1}$, $v_{2n+1} = \frac{1}{v_2}$. Thus $\frac{v_{2n+2}}{v_{2n+1}} = \frac{v_2}{v_1} > q$.
Statement 2 is proved.

Lemma 1 is proved.

Lemma 2. *Suppose for some $q > 1$ all zeros of $R_{2n+1}(y,q)$ are real and these zeros satisfy (7). Then all zeros of $R_{2n-1}(y,q)$ are real and these zeros satisfy (7).*

Proof of Lemma 2. The reality of zeros is a consequence of Theorem G (1). We have

$$\begin{gathered} R_{2n+1}(y,q) = \textstyle\sum_{k=0}^{2n-1}(-1)^k q^{k(2n+1-k)}y^k + q^{2n}y^{2n} - y^{2n+1} = \\ \textstyle\sum_{k=0}^{2n-1}(-1)^k q^{k(2n-1-k)}(q^2y)^k + q^{2n}y^{2n} - y^{2n+1} = \\ R_{2n-1}(q^2y, q) + q^{2n}y^{2n} - y^{2n+1}, \end{gathered} \tag{23}$$

or

$$R_{2n-1}(y,q) = R_{2n+1}(\frac{y}{q^2}, q) - \frac{y^{2n}}{q^{2n}} + \frac{y^{2n+1}}{q^{2(2n+1)}}. \tag{24}$$

Also we have

$$R_{2n+1}(y,q) = -q^{2n}yR_{2n-1}(y,q) + 1 - y^{2n+1}, \tag{25}$$

or

$$q^{2n}yR_{2n-1}(y,q) = -R_{2n+1}(y,q) + 1 - y^{2n+1}. \tag{26}$$

Since

$$R_{2n-1}(y,q) = -y^{2n-1}R_{2n-1}(\frac{1}{y}, q), \tag{27}$$

it is sufficient to prove the statement of Lemma for such zeros of $R_{2n-1}(y,q)$ that lie in the segment $[0,1]$. Denote by $y_1 < y_2 < \ldots < y_n < 1$ the first $n+1$ zeros of the polynomial $R_{2n+1}(y,q)$. Obviously, for every $k \in \mathbf{N}$ $R_{2k+1}(1,q) = 0$. By (24)

$$R_{2n-1}(y,q) < 0, \text{ for } y_{2k-1}q^2 < y < y_{2k}q^2,\ k = 1, 2, \ldots, \lfloor\frac{n+1}{2}\rfloor. \tag{28}$$

For $y \leq 1$ by (24) we have

$$R_{2n-1}(y,q) > 0, \text{ for } y_{2k-1} < y < y_{2k}, \ k = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor. \tag{29}$$

Since $S_{2n+1}(x,q) = R_{2n+1}(\frac{x}{q^{2n+1}}, q)$ and $S_{2n+1}(x,q)$ has exactly two zeros in the interval (q, q^3) we derive that $R_{2n+1}(y,q)$ has exactly two zeros in the interval $(\frac{1}{q^{2n}}, \frac{1}{q^{2n-2}})$. Moreover, $S_{2n-1}(x,q)$ does not have zeros in the interval $(0, q)$, so $R_{2n-1}(y,q)$ does not have zeros in the interval $(0, \frac{1}{q^{2n-2}})$. Consequently, $[y_1, y_2] \subset (0, \frac{1}{q^{2n-2}})$, and $R_{2n-1}(y,q) > 0$ for $y \in (0, \frac{1}{q^{2n-2}})$. Since $\frac{y_{2k+1}}{y_{2k-1}} > q^2$, we obtain

$$y_1 < y_2 < q^2 y_1 < q^2 y_2 < y_3 < y_4 < q^2 y_3 < q^2 y_4 < \dots$$
$$< y_{2\lfloor \frac{n+1}{2} \rfloor - 1} < y_{2\lfloor \frac{n+1}{2} \rfloor} < q^2 y_{2\lfloor \frac{n+1}{2} \rfloor - 1} < q^2 y_{2\lfloor \frac{n+1}{2} \rfloor}.$$

Suppose $n = 2m$. Denote by $w_1, w_2, \dots, w_{4m-1}$ the zeros of R_{2n-1} (we will consider the first $2m$ zeros of R_{2n+1} situated in the segment $[0,1]$, $w_{2m} = 1$). Note that $\frac{1}{y_{2m}} = \frac{y_{2m+1}}{y_{2m}} > q$, $\frac{1}{y_{2m-1}} = \frac{y_{2m+1}}{y_{2m}} \cdot \frac{y_{2m}}{y_{2m-1}} > q^2$. Using (28) and (29) we derive

$$\begin{gathered} y_1 < y_2 < w_1 < q^2 y_1 < q^2 y_2 < w_2 < y_3 < y_4 < w_3 < q^2 y_3 < q^2 y_4 \\ < \dots < w_{2m-1} < q^2 y_{2m-1} < q^2 y_{2m} < y_{2m+1} = w_{2m} = 1. \end{gathered} \tag{30}$$

As in the proof of lemma 1 we conclude that $\frac{w_{k+1}}{w_k} > q, \ k = 1, 2, \dots, 2m-1$, whence $\frac{w_{k+1}}{w_k} > q, \ k = 1, 2, \dots, 4m-2$.

Suppose $n = 2m+1$. Denote by $t_1, t_2, \dots, t_{4m+1}$ the zeros of R_{2n-1} (we will consider the first $2m+1$ zeros of R_{2n-1} situated in the segment $[0,1]$, $t_{2m+1} = 1$). Using (28) and (29) we derive

$$\begin{gathered} y_1 < y_2 < t_1 < q^2 y_1 < q^2 y_2 < t_2 < y_3 < y_4 < t_3 < q^2 y_3 < q^2 y_4 \\ < \dots < t_{2m-1} < q^2 y_{2m-1} < q^2 y_{2m} < t_{2m} < y_{2m+1} < y_{2m+2} = t_{2m+1} = 1. \end{gathered} \tag{31}$$

As in the proof of lemma 1 we conclude that $\frac{t_{k+1}}{t_k} > q, \ k = 1, 2, \dots, 2m$, whence $\frac{w_{k+1}}{w_k} > q, \ k = 1, 2, \dots, 4m$.

Lemma 2 is proved.

Suppose all zeros of $g_a(z)$ are real and simple (according to Theorem G (8) it means that $a > r_\infty$). Denote by $0 < x_1 < x_2 < \dots$ the zeros of $g_a(z)$.

Lemma 3. $\frac{x_{n+1}}{x_n} > \frac{x_2}{x_1}, \ n = 2, 3, 4, \dots$.

Proof of Lemma 3. Note that

$$\begin{gathered} g_a(x) = (1 - \frac{x}{a} + \dots + (-1)^{n-2} \frac{x^{n-2}}{a^{(n-2)^2}}) + \\ (-1)^{n-1} \frac{x^{n-1}}{a^{(n-1)^2}} (1 - \frac{x}{a^{2n-1}} + \dots + (-1)^{k-n+1} \frac{x^{k-n+1}}{a^{k^2-(n-1)^2}} + \dots) \\ = S_{n-2}(x,a) + (-1)^{n-1} \frac{x^{n-1}}{a^{(n-1)^2}} g_a(\frac{x}{a^{2n-2}}), \ n = 2, 3, \dots \end{gathered} \tag{32}$$

Let $x \in (x_1 a^{2n-2}, x_2 a^{2n-2})$. Then $g_a(\frac{x}{a^{2n-2}}) < 0$. Hence,

$$\operatorname{sign}\left((-1)^{n-1}\frac{x^{n-1}}{a^{(n-1)^2}}g_a(\frac{x}{a^{2n-2}})\right) = (-1)^{n-2},\ x \in (x_1 a^{2n-2}, x_2 a^{2n-2}). \tag{33}$$

By Theorem G (9), $a < x_1 < x_2 < a^3$, so $x_1 a^{2n-2} \geq a^{2n-1} > a^{2n-5}$ and by Theorem G (5)

$$(-1)^{n-2}S_{n-2}(x,a) > 0,\ x \in (x_1 a^{2n-2}, x_2 a^{2n-2}). \tag{34}$$

By (32), (33) and (34) we obtain that

$$(-1)^{n-2}g_a(x) > 0,\ x \in (x_1 a^{2n-2}, x_2 a^{2n-2}), n = 2, 3, \ldots. \tag{35}$$

Thus we have $2n$ intervals $(0, x_1)$, (x_1, x_2), $(x_1 a^2, x_2 a^2)$, $(x_1 a^4, x_2 a^4)$, ... , $(x_1 a^{4n-4}, x_2 a^{4n-4})$, contained in the interval $(0, a^{4n-1})$, and $g_a(x)$ is alternating in sign on these intervals. Using Theorem G (10) we deduce that

$$x_j < x_1 a^{2j-2} < x_2 a^{2j-2} < x_{j+1},\ j = 2, 3, \ldots, 2n-1.$$

Since $n \geq 2$ is arbitrary, we have

$$x_j < x_1 a^{2j-2} < x_2 a^{2j-2} < x_{j+1},\ j \geq 2. \tag{36}$$

Thus, $\frac{x_{n+1}}{x_n} > \frac{x_2}{x_1},\ n \geq 2$.
Lemma 3 is proved.

We recall that by (13) ($S_n(x,a)$ has all zeros in the right half plane $\Leftrightarrow a > s_n$, and that all zeros of $S_{2n+1}(x,a)$ lie in the right half plane if and only if all zeros of $R_n(y, a^4)$ are real and these zeros satisfy (7). The statement of Lemma 1 is equivalent to

$$s_{4n+3} \leq s_{4n+1}, \quad s_{4n+5} \leq s_{4n+1}, \quad n = 1, 2, \ldots. \tag{37}$$

The second of these inequalities gives

$$1.2106 \approx s_5 \geq s_9 \geq s_{13} \geq \ldots. \tag{38}$$

Thus there exists a limit $\lim_{n\to\infty} s_{4n+1}$ and

$$B := \lim_{n\to\infty} s_{4n+1}. \tag{39}$$

The statement of Lemma 2 is equivalent to

$$s_{4n+3} \geq s_{4n-1}, \quad n = 1, 2, \ldots, \tag{40}$$

or,

$$1 = s_3 \leq s_7 \leq s_{11} \leq \ldots. \tag{41}$$

Thus there exists a limit $\lim_{n\to\infty} s_{4n+3}$ and

$$C := \lim_{n\to\infty} s_{4n+3}. \tag{42}$$

By (37) we have

$$B \geq C. \tag{43}$$

It is well-known (see [12, Chapter 7]) that the Hermite-Biehler Criterion is valid not only for polynomials, but for some classes of entire functions. The following statement is a generalization of the Hermite-Biehler theorem on entire functions of order not greater then 1 and minimal type of growth.

Theorem (B.Ja.Levin). *Let* $G(z) = \sum_{k=0}^{\infty} a_k z^k, a_k > 0$, *be an entire function of order not greater then 1 and minimal type of growth. All zeros of the entire function* G *have negative real parts if and only if the following two entire functions* $f(z) = \sum_0^\infty (-1)^m a_{2m} z^m$ *and* $g(z) = z\sum_0^\infty (-1)^m a_{2m+1} z^m$ *have simple real interlacing zeros.*

It is well-known that the order of an entire function with coefficients a_k is given by the expression $\limsup_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}}$ (see, for example [12, Chapter 1]). So, the entire function $g_a(z)$ is of order 0 and thus the Hermite-Biehler Criterion is valid for this function. It is easy to show that (analogously to the polynomial case) if for some $a_0 > 1$ all the zeros of the entire function $g_{a_0}(z)$ are situated in the right half-plane then for all $a > a_0$ all the zeros of the entire function $g_a(z)$ are situated in the right half-plane. Thus

$$\exists s_\infty > 1 : \text{ (the zeros of } g_a(z) \text{ are in the right half-plane } \Leftrightarrow a > s_\infty). \tag{44}$$

Let us prove that $s_\infty \leq C$. Note that $\lim_{n\to\infty} S_{4n+3}(x, a) = g_a(x)$, and this limit is uniform on the compact sets. This implies that for all $a \geq C$ all the zeros of g_a lie in the closed half-plane $\{z : \mathrm{Re} z \geq 0\}$. Suppose that $C < s_\infty$. Then for all $a \in (C; s_\infty]$ the function g_a has all zeros in the closed right half-plane, but not in the open right half-plane. It means that for all $a \in (C; s_\infty]$ the function g_a has an imaginary zero $ix_0(a), x_0(a) \in \mathbf{R}$. Then

$$1 - \frac{x_0(a)^2}{a^4} + \frac{x_0(a)^4}{a^{16}} - \frac{x_0(a)^6}{a^{36}} + \cdots = 0$$

and

$$1 - \frac{x_0(a)^2}{a^8} + \frac{x_0(a)^4}{a^{24}} - \frac{x_0(a)^6}{a^{48}} + \cdots = 0.$$

Denote by $t_0(a) = x_0(a)^2, q = a^4$. Then

$$g_q(t_0(a)) = 1 - \frac{t_0(a)}{q} + \frac{t_0(a)^2}{q^4} - \frac{t_0(a)^3}{q^9} + \cdots = 0 \tag{45}$$

and

$$g_q(\frac{t_0(a)}{q}) = 1 - \frac{t_0(a)}{q^2} + \frac{t_0(a)^2}{q^6} - \frac{t_0(a)^3}{q^{12}} + \cdots = 0. \tag{46}$$

Let us denote by $x_1(q) < x_2(q) < x_3(q) < \ldots$ the zeros of the function $g_q(t)$. By (45) and (46) there exists $j \in \mathbb{N}$ such that $\frac{t_0(a)}{q} = x_j(q),\ t_0(a) = x_{j+1}$. So for all $q \in (C^4; s_\infty^4]$ there exists $j \in \mathbf{N}$ such that

$$\frac{x_{j+1(q)}}{x_j(q)} = q. \tag{47}$$

Since $a > C$ all the zeros of the polynomial $S_{4n+3}(x, a)$ lie in the open right half-plane. It means that for $q = a^4$ all the zeros of the polynomial $P_{2n+1}(t, q) = 1 - \frac{t}{q} + \frac{t^2}{q^4} - \ldots + (-1)^{2n+1} \frac{t^{2n+1}}{q^{(2n+1)^2}}$ denoted by $t_1(q, 2n+1), t_2(q, 2n+1), \ldots, t_{2n+1}(q, 2n+1)$ are real and distinct. Moreover as we have proved these zeros satisfy the condition $\frac{t_{j+1}(q,2n+1)}{t_j(q,2n+1)} > q$ for $n \in \mathbf{N},\ j = 1, 2, \ldots, 2n$, and $q = a^4 > C^4$. Since $\lim_{n\to\infty} t_j(q, 2n+1) = x_j(q)$ we obtain $\frac{x_{j+1}(q)}{x_j(q)} \geq q,\ q > C^4$. From this and (47) we obtain using Lemma 3 that $\frac{x_2(q)}{x_1(q)} = q,\ q \in (C^4; s_\infty^4]$, or

$$x_2(q) - q x_1(q) = 0,\ q \in (C^4; s_\infty^4]. \tag{48}$$

Let us show that (48) is impossible. We will use the following variant of the well-known theorem on the regularity of implicit function.

Theorem H. ([7, Chapter 2]) *Let a complex function $F(q, z)$ of two complex variables is jointly continuous in a neighborhood of a point (q_0, z_0) and is regular in each variable in the same neighborhood. If $F(q_0, z_0) = 0$ and $F_z'(q_0, z_0) \neq 0$ then there exist positive numbers ε, δ such that for every $q, |q - q_0| < \varepsilon$, the equation $F(q, z) = 0$ has the unique solution $z = \varphi(q)$ under condition $|z - z_0| < \delta$. And at that the function $z = \varphi(q)$ is regular at the point q_0.*

By (14) and by the fifth and the seventh inequalities of (37) we conclude that $1.16 < s_\infty < 1.21$. So $s_\infty^4 > r_\infty$, where r_∞ is the constant from Theorem J. Thus by Theorem J (8) the function $g_q(t)$ does not have multiple zeros for $q \geq s_\infty$. Hence we can apply Theorem H to the function $g_q(t)$ at the point $(s_\infty^4, x_1(s_\infty^4))$ and at the point $(s_\infty^4, x_2(s_\infty^4))$. So there exist positive numbers ε, δ such that for every $q, |q - s_\infty^4| < \varepsilon$, the equation $g_q(t) = 0$ has the unique solution $x_1 = x_1(q)$ satisfying the condition $|x_1(q) - x_1(s_\infty^4)| < \delta$ and the equation $g_q(t) = 0$ has the unique solution $x_2 = x_2(q)$ satisfying the condition $|x_2(q) - x_2(s_\infty^4)| < \delta$. Moreover both functions $x_1 = x_1(q)$ and $x_2 = x_2(q)$ are regular in some circle $U_\rho(s_\infty^4) = \{q : |q - s_\infty^4| < \rho\}, \rho > 0$. Whence the function $x_2(q) - q x_1(q)$ is regular in the circle $U_\rho(s_\infty^4)$. By (48) we have $x_2(q) - q x_1(q) = 0$ for $q \in (C^4; s_\infty^4] \cap U_\rho(s_\infty^4)$. By the uniqueness theorem it means that $x_2(q) - q x_1(q) \equiv 0$ for $q \in U_\rho(s_\infty^4)$, but this conclusion contradicts the condition $\frac{x_2(q)}{x_1(q)} > q$ for $q > s_\infty^4$. This contradiction shows that $s_\infty \leq C$. Uniting with (43) we have

$$s_\infty \leq C \leq B. \tag{49}$$

It remains to prove that $s_\infty \geq B$. Suppose that $s_\infty < B$. Let us choose $a, s_\infty < a < B$, and denote by $q = a^4$. By the B.Ja.Levin's theorem all the zeros of

$g_q(t) = 1 - \frac{t}{q} + \frac{t^2}{q^4} - \frac{t^3}{q^9} + \cdots$ and $g_q(\frac{t}{q}) = 1 - \frac{t}{q^2} + \frac{t^2}{q^6} - \frac{t^3}{q^{12}} + \cdots$ are real and interlacing. If $x_1 < x_2 < x_3 < \cdots$ be the real zeros of $g_q(t)$, then, as we have shown, there exists $\varepsilon > 0$ such that

$$\forall j = 1, 2, 3, \dots \ \frac{x_{j+1}}{x_j} > q + \varepsilon. \tag{50}$$

We have

$$\tilde{P}_{2n}(t, q) = 1 - \frac{t}{q} + \frac{t^2}{q^4} \ - \ldots + \frac{t^{2n}}{q^{(2n)^2}} = g_q(t) + \frac{t^{2n+1}}{q^{(2n+1)^2}} - \frac{t^{2n+2}}{q^{(2n+2)^2}} + \ldots.$$

Hence, if $x_{2k} \leq t \leq x_{2k+1}, \ k = 1, 2, \dots, n-1$, then $\tilde{P}_{2n}(t, q) > 0$. If we denote by $t_k(2n), \ 1 \leq k \leq 2n$, the zeros of $\tilde{P}_{2n}(t, q)$, then $t_{2k}(2n) < x_{2k} < x_{2k+1} < t_{2k+1}(2n)$, i.e.

$$\frac{t_{2k+1}(2n)}{t_{2k}(2n)} > q + \varepsilon, \ k = 1, 2, 3, \dots \ n-1. \tag{51}$$

Since $\tilde{P}_{2n}(t, q) \rightrightarrows g_q(t), \ n \to \infty$, on the compact sets, we have $t_1(2n) \to x_1, \ t_2(2n) \to x_2$. Hence,

$$\exists n_0 \ \forall n \geq n_0 : \frac{t_2(2n)}{t_1(2n)} \geq q + \frac{\varepsilon}{2}. \tag{52}$$

Lemma 4. *The inequality* $\frac{t_{2k}(4n-4)}{t_{2k-1}(4n-4)} > \frac{t_2(2n)}{t_1(2n)}$ *holds for* $n = 3, 4, 5, \dots$ *and* $k = 1, 2, \dots, 2n-2$.

Proof of Lemma 4. We have

$$\tilde{P}_{2n+2}(t, q) = \tilde{P}_{2n}(t, q) - \frac{t^{2n+1}}{q^{(2n+1)^2}} + \frac{t^{2n+2}}{q^{(2n+2)^2}}$$

and

$$\tilde{P}_{2n+2}(t, q) = 1 - \frac{t}{q} + \frac{t^2}{q^4} \tilde{P}_{2n}(\frac{t}{q^4}, q).$$

Therefore if $t_1(2n) < t < t_2(2n)$ or $t_1(2n)q^4 < t < t_2(2n)q^4$ we have $\tilde{P}_{2n+2}(t, q) < 0$. Whence

$$\frac{t_2(2n+2)}{t_1(2n+2)} > \frac{t_2(2n)}{t_1(2n)}, \quad \frac{t_4(2n+2)}{t_3(2n+2)} > \frac{t_2(2n)}{t_1(2n)}. \tag{53}$$

Applying this reasoning to $\tilde{P}_{2n+4}(t, q)$ we obtain from (53)

$$\frac{t_2(2n+4)}{t_1(2n+4)} > \frac{t_2(2n)}{t_1(2n)}, \quad \frac{t_4(2n+4)}{t_3(2n+4)} > \frac{t_2(2n)}{t_1(2n)}, \quad \frac{t_6(2n+4)}{t_5(2n+4)} > \frac{t_2(2n)}{t_1(2n)}, \tag{54}$$

and so on. For $\tilde{P}_{4n-4}(t, q)$ we have

$$\frac{t_{2k}(4n-4)}{t_{2k-1}(4n-4)} > \frac{t_2(2n)}{t_1(2n)}, \quad k = 1, 2, \dots, n-1. \tag{55}$$

Note that

$$\tilde{P}_{4n-4}(t,q) = \frac{t^{4n-4}}{q^{(4n-4)^2}} \tilde{P}_{4n-4}(\frac{q^{4n-6}}{t}, q).$$

Therefore,

$$t_j(4n-4) = \frac{q^{4n-6}}{t_{4n-4-j+1}(4n-4)} = \frac{q^{4n-6}}{t_{4n-j-3}(4n-4)}.$$

Whence using (55) we obtain

$$\frac{t_{2k}(4n-4)}{t_{2k-1}(4n-4)} = \frac{t_{4n-2k-2}(4n-4)}{t_{4n-2k-3}(4n-4)} > \frac{t_2(2n)}{t_1(2n)}, \quad 2n-k-1 = 1, 2, \dots, n-1,$$

or

$$\frac{t_{2k}(4n-4)}{t_{2k-1}(4n-4)} = \frac{t_{4n-2k-2}(4n-4)}{t_{4n-2k-3}(4n-4)} > \frac{t_2(2n)}{t_1(2n)}, \quad k = n, n+1, \dots, 2n-2.$$

Taking into account (55) we obtain the statement of lemma.

Lemma 4 is proved.

Let us fix an integer $m \geq n_0$ in (52). By lemma 4 we have

$$\frac{t_{2k}(4m-4)}{t_{2k-1}(4m-4)} > q + \frac{\varepsilon}{2}, \quad k = 1, 2, \dots, 2m-2.$$

Using (51) we obtain

$$\frac{t_{2k+1}(4m-4)}{t_{2k}(4m-4)} \geq q + \varepsilon > q + \frac{\varepsilon}{2}, \quad k = 1, 2, \dots, 2m-3.$$

Hence for all zeros of $\tilde{P}_{4nm-4}(t,q)$ the inequality holds

$$\frac{t_{j+1}(4m-4)}{t_j(4m-4)} > q + \frac{\varepsilon}{2} > q, j = 1, 2, \dots, 4m-5.$$

This means that all zeros of $S_{8m-7}(x,a)$ lie in the open right half-plane for $s_\infty < a < B$ and for all $m \geq n_0$. This contradicts the definition of the constant B (see (39). Thus,

$$s_\infty \geq B. \tag{56}$$

Uniting with (49) finally we get $B = C = s_\infty$.

Theorem 1 is proved.

Acknowledgement. The authors are deeply grateful to the referee for valuable remarks and suggestions.

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